15. (a) State and prove: The Schwarz inequality.

Or

(b) State and prove: The minkowski inequality.

SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. State and prove Riemann-Lebesgue lemma.

17. Assume that g is differentiable at a. with total derivative $g'(a)$.Let $b = g(a)$ and assume that f is differentiable at b, with total derivative $f'(b)$. Show that the composite function $h = f^{\circ} g$ is

differentiable at a. and the total derivative $h'(a)$ is given by

$h'(a) = f'(b)^{\circ} g'(a)$,

The composition of the linear function $f'(b)$ and $g'(a)$.

18. State and prove the Implicit function theorem.

19. If E_1 and E_2 are subsets of [a, b] prove that $\overline{m}\, E_1 + \overline{m}\, E_2 \geq \overline{m}(E_1 \cup E_2) + \overline{m}\,(E_1 \cap E_2)$ and $\underline{m}\, E_1 + \underline{m} E_2 \leq \underline{m}(E_1 \cup E_2) + \underline{m}(E_1 \cap E_2)$.

20. Show that the metric space $L^2[a,b]$ is complete.

APRIL/MAY 2024

DMA22 — REAL ANALYSIS II

Time : Three hours Maximum : 75 marks

Shanmuga Industries Arts & Science College T.V.Malai

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

1. State Jordan's test.

2. Define Fourier series of f relative to s.

3. If f is differential at c, then f is continuous at c.

4. Define matrix form of the chain rule.

5. Write short on Jacobian determinant.

6. Define open mapping.

7. Define: Inner and Outer Measure.

8. Define: Measurable function.

9. Define: The lebesgue integral for unbounded functions.

10. Define: Metric space $L^2[a,b]$.

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions.

11. (a) If $f \in L(-\infty, +\infty)$ prove that

$$\lim_{\alpha \to +\infty} \int_{-\infty}^{\infty} f(t) \frac{1-\cos \alpha t}{t} dt = \int_{0}^{\infty} \frac{f(t)-f(-t)}{t} dt,$$

Whenever the Lebesgue integral on the right exists.

Or

(b) If g is of bounded variation on $[0, \delta]$, show that.

$$\lim_{\alpha \to +\infty} \frac{2}{\pi} \int_{0}^{\delta} g(t) \frac{\sin \alpha t}{t} dt = g(0+).$$

12. (a) State and prove the Mean-value theorem.

Or

(b) Let $f : s \to R^m$ be differentiable at an interior point c of S, where $S \subseteq R^n$. If $v = v_1 u_1 + \cdots + v_n u_n$, where $u_1, \cdots u_n$ are the unit coordinate vector in R^n, prove that

$$f'(c)(v) = \sum_{k=1}^{n} v_k D_k f(c).$$

In particular, if f is a real-valued ($m = 1$) we have $f'(c)(v) = \nabla f(c) \cdot v$, the dot product of v with the vector $\nabla f(c) = (D_1 f(c), \cdots D_n f(c))$.

13. (a) Let f be a real - valued function with continuous second-order partial derivatives at a stationary point a in R^2. Let.

$$A = D_{1,1} f(a), B = D_{1,2} f(a), C = D_{2,2} f(a)$$

$\Delta = \det \begin{bmatrix} A & B \\ B & C \end{bmatrix} = AC - B^2$. Prove that.

(i) If $\Delta > 0$ and $A > 0$, f has a relative minimum at a.

(ii) If $\Delta > 0$ and $A < 0$, f has a relative maximum at a.

(iii) If $\Delta < 0$, f has a saddle point at a.

Or

(b) If $f = u + iv$ is a complex-valued function with a derivative at a point z in C, show that $J_f(z) = |f'(z)|^2$.

14. (a) If $E_1, E_2, \cdots$ are pairwise disjoint measurable subsets of [a,b] show that $\bigcup_{n=1}^{\infty} E_n$ is measurable and $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} mE_n$.

Or

(b) If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable function on [a, b] and if $\lim_{n \to \infty} f_{n(x)} = f(x)$. almost everywhere $(a \le x \le b)$ prove that measurable.